

# LEBESGUE MEASURABILITY OF SEPARATELY CONTINUOUS FUNCTIONS AND SEPARABILITY

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**ABSTRACT.** It is studied a connection between the separability and the countable chain condition of spaces with the  $L$ -property (a topological space  $X$  has the  $L$ -property if for every topological space  $Y$ , separately continuous function  $f : X \times Y \rightarrow \mathbb{R}$  and open set  $I \subseteq \mathbb{R}$  the set  $f^{-1}(I)$  is a  $F_\sigma$ -set). We show that every completely regular Baire space with the  $L$ -property and the countable chain condition is separable and construct a nonseparable completely regular space with the  $L$ -property and the countable chain condition. This gives a negative answer to a question of M. Burke.

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2000 Mathematics Subject Classification 54C05, 54C10, 54E52

## 1. INTRODUCTION

A function  $f : X \rightarrow \mathbb{R}$  defined on a topological space  $X$  is called a *first Baire class function* if there exists a sequence  $(f_n)_{n=1}^\infty$  of continuous functions  $f_n : X \rightarrow \mathbb{R}$  which converges pointwise to  $f$  on  $X$ ; and a *first Lebesgue class function* if  $f^{-1}(G)$  is a  $F_\sigma$ -set for every open set  $G \subseteq \mathbb{R}$ . Standard reasons (see [1, p. 394]) shows that every first Baire class function is a first Lebesgue class function.

Investigations of Baire and Lebesgue classifications of separately continuous functions were started by H. Lebesgue in [2] and were continued in papers of many mathematicians (see [3]).

We say that a topological space  $X$  has the *B-property* (the *L-property*) if for every topological space  $Y$  each separately continuous function  $f : X \times Y \rightarrow \mathbb{R}$  is a first Baire class function (a first Lebesgue class function).

It is known [4, 5] that any topological space  $X$  has the  $B$ -property (the  $L$ -property) if and only if the evaluation function  $c_X : X \times C_p(X) \rightarrow \mathbb{R}$ ,  $c_X(x, y) = y(x)$ , is a first Baire class function (a first Lebesgue class function), where  $C_p(X)$  means the space of continuous on  $X$  functions with the pointwise convergence topology.

Baire and Lebesgue classifications of separately continuous function were investigated in [6]. In particular, it was shown in [6] that any completely

regular space  $X$  with the  $B$ -property and the countable chain condition is separable (topological space  $X$  has a countable chain condition (C.C.C.) if every system of disjoint open in  $X$  sets is at most countable). In this connection the following question arose in [6, Problem 4.6].

**Question 1.1.** *Is every completely regular space  $X$  with the  $L$ -property and the countable chain condition a separable space?*

In this paper we show that if a space  $X$  is a Baire space then Question 1.1 has a positive answer and construct an example which gives a negative answer to the question in general case.

## 2. DENSITY OF BAIRE SPACES WITH THE $L$ -PROPERTY

The minimal cardinal  $\aleph \geq \aleph_0$  for which any system of disjoint open in a topological space  $X$  sets has the cardinality at most  $\aleph$  is called a *Souslin number of  $X$*  and is denoted by  $c(X)$ . Note that the countable chain condition of  $X$  means that  $c(X) = \aleph_0$ . It is easy to see that  $c(X) \leq d(X)$ , where  $d(X)$  is the density of  $X$ .

The following result implies that for a Baire space  $X$  Question 1.1 has a positive answer.

**Theorem 2.1.** *Let  $X$  be a completely regular Baire space with the  $L$ -property. Then  $c(X) = d(X)$ .*

*Proof.* Since the evaluation function  $c_X$  is a first Lebesgue class function, the set  $E = \{(x, y) : y(x) = 0\}$  is a  $G_\delta$ -set in  $X \times Y$ , where  $Y = C_p(X)$ . Choose a sequence  $(W_n)_{n=1}^\infty$  of open in  $X \times Y$  sets  $W_n$  such that  $E = \bigcap_{n=1}^\infty W_n$ . Denote by  $y_0$  the null-function on  $Y$ . For every  $n \in \mathbb{N}$  and an  $x \in X$  find open neighborhoods  $U(x, n)$  and  $V(x, n)$  of  $x$  and  $y_0$  in  $X$  and  $Y$  respectively such that  $U(x, n) \times V(x, n) \subseteq W_n$ .

Fix a  $n \in \mathbb{N}$  and show that there exists a set  $A_n \subseteq X$  with  $|A_n| \leq c(X) = \aleph$  such that the open set  $G_n = \bigcup_{x \in A_n} U(x, n)$  is dense in  $X$ . Consider a system  $\mathcal{U}$  of all open in  $X$  nonempty sets  $U$  such that  $U \subseteq U(x, n)$  for some  $x \in X$  and choose a maximal system  $\mathcal{U}' \subseteq \mathcal{U}$  which consists of disjoint sets. It is clear that  $|\mathcal{U}'| \leq \aleph$ . For every  $U \in \mathcal{U}'$  find an  $x = x(U) \in X$  such that  $U \subseteq U(x, n)$  and put  $A_n = \{x(U) : U \in \mathcal{U}'\}$ . Then  $|A_n| \leq |\mathcal{U}'| \leq \aleph$ . Besides, it follows from the maximality of  $\mathcal{U}'$  that  $G_n$  is dense in  $X$ .

Since  $X$  is a Baire space, the set  $X_0 = \bigcap_{n=1}^\infty G_n$  is dense in  $X$ . For every  $n \in \mathbb{N}$  and  $x \in X$  choose a finite set  $B(x, n) \subseteq X$  such that  $y \in V(x, n)$  for each  $y \in Y$  with  $y|_{B(x, n)} = y_0|_{B(x, n)}$ . Put  $B = \bigcup_{n \in \mathbb{N}} \bigcup_{x \in A_n} B(x, n)$ . Note that  $|B| \leq \aleph_0 \cdot \aleph = \aleph$ .

Show that  $B$  is dense in  $X$ . Since  $X$  is a completely regular space it is enough to prove that  $y_0$  is a unique continuous on  $X$  function which equals to 0 at every point from  $B$ . Let  $y \in Y$  be a function such that  $y(b) = 0$  for every  $b \in B$ . Fix a point  $x \in X_0$  and an integer  $n \in \mathbb{N}$ . Find  $a \in A_n$  such that  $x \in U(a, n)$ . Then  $B(a, n) \subseteq B$  implies  $y \in V(a, n)$ . Therefore  $(x, y) \in W_n$ . Thus  $X_0 \times \{y\} \subseteq \bigcap_{n=1}^{\infty} W_n = E$ , that is  $y(x) = 0$  for every  $x \in X_0$ . Hence  $y = y_0$  because  $X_0$  is dense in  $X$ .

Thus  $d(X) \leq |B| \leq c(X)$ . Therefore  $c(X) = d(X)$ . □

**Corollary 2.2.** *Every completely regular Baire space with the  $L$ -property and the countable chain condition is a separable space.*

### 3. NONSEPARABLE SPACES WITH THE $L$ -PROPERTY AND C.C.C.

The following notion was introduced in [4], where some properties of spaces with the  $B$ -property were studied.

A topological space  $X$  with a topology  $\tau$  is called *quarter-stratifiable* if there exists a function  $g : \mathbb{N} \times X \rightarrow \tau$  such that

- (i)  $X = \bigcup_{x \in X} g(n, x)$  for every  $n \in \mathbb{N}$ ;
- (ii) if  $x \in g(n, x_n)$  for each  $n \in \mathbb{N}$  then  $x_n \rightarrow x$ .

The following result follows from [7, Proposition 2.1].

**Proposition 3.1.** *Every quarter-stratifiable space  $X$  has the  $L$ -property.*

A topological space  $X$  is called  $\sigma$ -discrete if there exists an increasing sequence  $(X_n)_{n=1}^{\infty}$  of closed discrete subspaces  $X_n$  of  $X$  such that  $X = \bigcup_{n=1}^{\infty} X_n$ .

**Proposition 3.2.** *Every  $\sigma$ -discrete space is a quarter-stratifiable space.*

*Proof.* Let  $(X_n)_{n=1}^{\infty}$  be an increasing sequence of closed discrete subspaces  $X_n$  of  $X$  such that  $X = \bigcup_{n=1}^{\infty} X_n$ . For every  $n \in \mathbb{N}$  and  $x \in X_n$  denote by  $U(x, n)$  an open in  $X$  neighborhood of  $x$  such that  $U(x, n) \cap X_n = \{x\}$ . A function  $g : \mathbb{N} \times X \rightarrow \tau$ , where  $\tau$  is the topology of  $X$ , define letting:  $g(x, n) = U(x, n)$  if  $x \in X_n$  and  $g(x, n) = X \setminus X_n$  if  $x \notin X_n$ . It is easy to see that  $g$  satisfies (i) and (ii). □

Show now that Question 1.1 has a negative answer.

**Theorem 3.3.** *There exists a completely regular nonseparable space with the  $L$ -property and with the countable chain condition.*

*Proof.* Let  $\Gamma_0$  be a set with  $|\Gamma_0| \geq \aleph_1$ ,  $(a_n)_{n=1}^{\infty}$  be a sequence of distinct points  $a_n \notin \Gamma_0$ ,  $\Gamma_n = \Gamma_0 \cup \{a_k : 1 \leq k \leq n\}$ ,  $\mathcal{A}_n$  be a system of all subsets  $A \subseteq \Gamma_{n-1}$  such that  $|A| = n$ . Denote by  $X_n$  a set of all function  $x \in \{0, 1\}^{\Gamma}$  such that

$x = \chi_{A \cup \{a_n\}}$  for some  $A \in \mathcal{A}_n$ , where  $\chi_B$  means the characteristic function of  $B$ , and put  $X = \bigcup_{n=1}^{\infty} X_n$ .

Show that  $X$  is a  $\sigma$ -discrete space. For every  $n \in \mathbb{N}$  put  $Y_n = \bigcup_{k=1}^n X_k$ . Fix an integer  $n \in \mathbb{N}$  and for each  $1 \leq k \leq n$  put  $G_k = \{x \in X : x(a_k) = 1, x(a_i) = 0, k < i \leq n\}$ . It is easy to see that  $G_k \cap Y_n = X_k$ . Since all spaces  $X_k$  are discrete,  $Y_n$  is discrete in  $X$  too. Besides,  $Y_n$  is closed in  $X$ . Thus,  $X$  has the  $L$ -property by Propositions 3.1 and 3.2.

Note that  $X$  is dense in  $Y = \{0, 1\}^{\Gamma}$ . Indeed, let  $A \subseteq \Gamma$  be a finite set and  $y : A \rightarrow \{0, 1\}$ . Choosing  $n \geq |A|$  with  $A \subseteq \Gamma_n$  find  $x \in X_{n+1}$  such that  $x|_A = y$ . Then  $c(X) = \aleph_0$  since  $c(Y) = \aleph_0$  and  $X$  is dense in  $Y$ .

It remains to note that  $X$  is nonseparable because for every separable subspace  $Z$  of  $X$  there exists a countable set  $B \subseteq \Gamma$  such that  $z(\gamma) = 0$  for every  $\gamma \in \Gamma \setminus B$ .  $\square$

This example shows that there exists a quarter-stratifiable space which has not the  $B$ -property. Thus, Proposition 3.1 cannot be generalized for spaces with the  $B$ -property.

A family  $(A_i : i \in I)$  of sets  $A_i$  is called *pointwise finite* if  $\bigcap_{i \in J} A_i = \emptyset$  for each infinite set  $J \subseteq I$ . A cardinal

$p(X) = \sup\{|\mathcal{A}| : \mathcal{A} \text{ is a pointwise finite family of nonempty open in } X \text{ sets}\}$  is called a *point-finite cellularity of a topological space*  $X$ . Clearly  $c(X) \leq p(X)$ . Besides, it is known that  $p(X) = c(X)$  for each Baire space  $X$ . Therefore the following question arises naturally from Theorem 2.1 and the fact that  $p(X) = |\Gamma| > \aleph_0$  for the space  $X$  from Theorem 3.3.

**Question 3.4.** *Is every completely regular space  $X$  with the  $L$ -property and  $p(X) = \aleph_0$  a separable space?*

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